Stability and boundary stabilization of physical networks represented by 1-D hyperbolic balance laws

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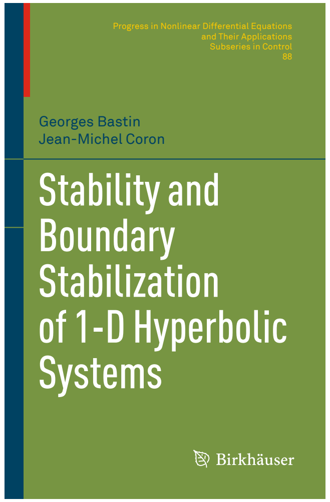
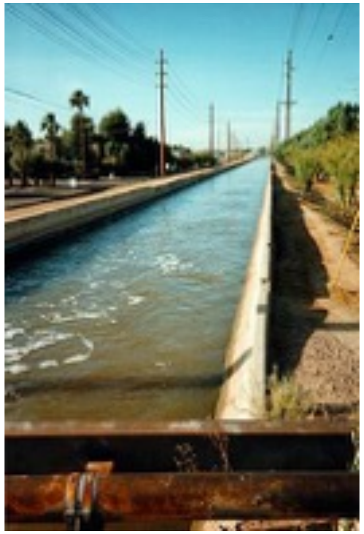


Outline

- Hyperbolic 2x2 systems of balance laws
- Steady-state and characteristic form
- Networks of balance laws
- Boundary conditions
- Exponential stability
- Lyapunov stability analysis
- Real-life application : hydraulic networks
- Non uniform systems



Open channels, St Venant equations (1871)

$$\partial_t h + \partial_x (hv) = 0$$

$$\partial_t v + \partial_x (gh + \frac{1}{2} v^2) = gS - Cv^2/h$$

h = water level, v = water velocity,
g= gravity, S = canal slope, C = friction coefficient

Road traffic, Aw-Rascle equations (2000)

$$\partial_t \rho + \partial_x (\rho v) = 0$$

$$\partial_t (v + p(\rho)) + v \partial_x \big(v + p(\rho)\big) = (V(\rho) - v)/\tau$$

ρ = traffic density, v = traffic velocity,
$p(\rho)$ = "traffic pressure" , $V(\rho)$ = preferential velocity,
τ = time constant

Telegrapher equations , Heaviside (1892)

$$\partial_t(Li) + \partial_x v = -Ri,$$
$$\partial_t(Cv) + \partial_x i = -Gv,$$

i = current intensity, v = voltage, L = line inductance, C = capacitance, R = line resistance, G = dielectric admittance (per unit length).

Optical fibers : Raman amplifiers (1927)

$$\partial_t S + \lambda_s \partial_x S = \lambda_s \Big(-\alpha_s S + \beta_s SP \Big),$$
$$\partial_t P - \lambda_p \partial_x P = \lambda_p \Big(-\alpha_p P - \beta_p SP \Big),$$

$S(t,x)$ = transmitted signal power, $P(t,x)$ = pump laser beam power, λ_s and λ_p = propagation velocities, α_s and α_p = attenuation coefficients, β_s and β_p = amplification gains

Euler isentropic equations (1757)

$$\partial_t \rho + \partial_x(\rho v) = 0$$
$$\partial_t(\rho v) + \partial_x(\rho v^2 + P(\rho)) = -C\rho v|v|$$

ρ = gas density, v = gas velocity,
P = pressure, C = friction coefficient

Fluid flow in elastic tubes (e.g. blood flow)

$$\partial_t A + \partial_x(AV) = 0,$$
$$\partial_t(AV) + \partial_x\left(\alpha AV^2 + \kappa A^2\right) = -CV$$

A = tube cross-section, V = fluid velocity,
α, κ, C = constant coefficients

Hyperbolic 2x2 systems of balance laws

Space $x \in [0, L]$

Time $t \in [0, +\infty)$

State
$$Y(t,x) \triangleq \begin{pmatrix} y_1(t,x) \\ y_2(t,x) \end{pmatrix}$$

$$\partial_t Y + \partial_x f(Y) = g(Y)$$

$$\boxed{\partial_t Y + A(Y)\partial_x Y = g(Y)}$$

$A(Y)$ has 2 distinct real eigenvalues

Uniform steady state $\partial_t Y + A(Y)\partial_x Y = g(Y)$

A steady-state is a constant solution $Y(t,x) \equiv \bar{Y}$

which satisfies the equation $g(\bar{Y}) = 0$

and (obviously) the state equation $\partial_t \bar{Y} + A(\bar{Y})\partial_x \bar{Y} = g(\bar{Y})$

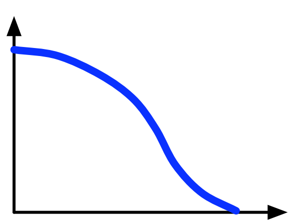
Road traffic

$\rho =$ density

$v =$ velocity

Steady state

$\bar{v} = V(\bar{\rho})$



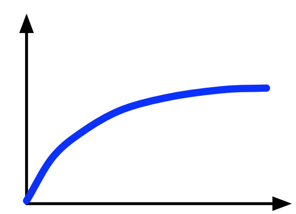
Open channels

$h =$ water depth

$v =$ velocity

Steady state

$$\bar{v} = \sqrt{\frac{gS}{C}\bar{h}}$$



(Toricelli formula)

Characteristic form

- Hyperbolic system : $\partial_t Y + A(Y)\partial_x Y = g(Y)$

$$\Downarrow$$

- Change of coordinates :

$$\xi(Y) = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \qquad \partial_t \xi + \begin{pmatrix} c_1(\xi) & 0 \\ 0 & c_2(\xi) \end{pmatrix} \partial_x \xi = h(\xi)$$

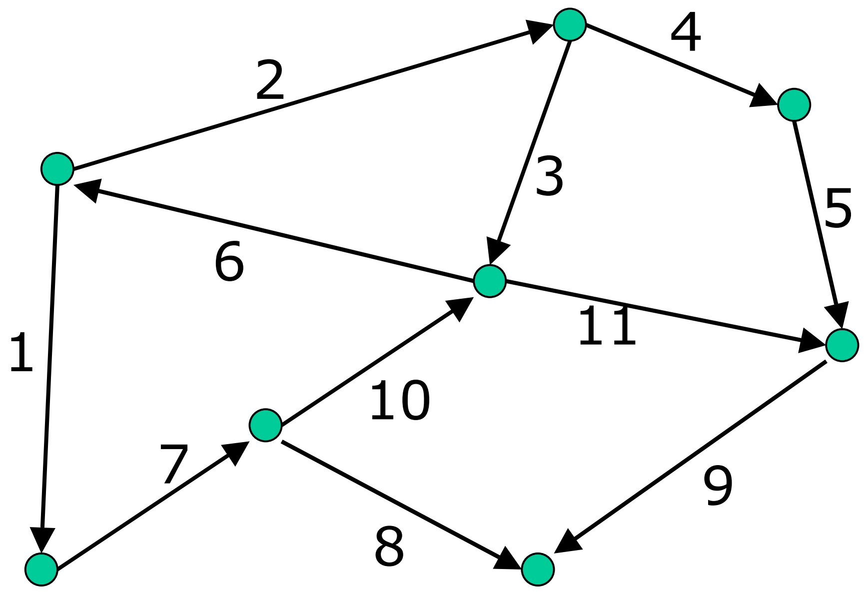
(Riemann coordinates)

with $c_1(\xi) \neq c_2(\xi)$ eigenvalues of $A(Y)$ (Characteristic velocities)

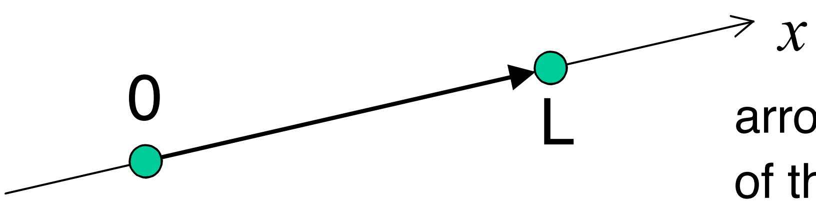
The change of coordinates $\xi(Y)$ is defined up to a constant.
It can therefore be selected such that $\xi(\bar{Y}) = 0 \quad \Rightarrow h(0) = 0.$

Physical netwoks of 2x2 hyperbolic systems

(e.g. hydraulic networks (irrigation, waterways) or road traffic networks)



- directed graph
- n edges
- one 2x2 hyperbolic system of balance laws attached to each arc



arrow = orientation of the x axis

$$\partial_t \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} + \begin{pmatrix} c_i(\xi) & 0 \\ 0 & c_{n+i}(\xi) \end{pmatrix} \partial_x \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} = h \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} \qquad (i = 1, \ldots, n)$$

Characteristic form

$$\partial_t \xi + C(\xi)\partial_x \xi = h(\xi)$$

(characteristic velocities)

$$C(\xi) = \mathrm{diag}(c_1(\xi), \dots, c_{2n}(\xi))$$

$$c_i(\xi) > 0 \quad i = 1, \dots, n$$

$$c_i(\xi) < 0 \quad i = n+1, \dots, 2n$$

Notations

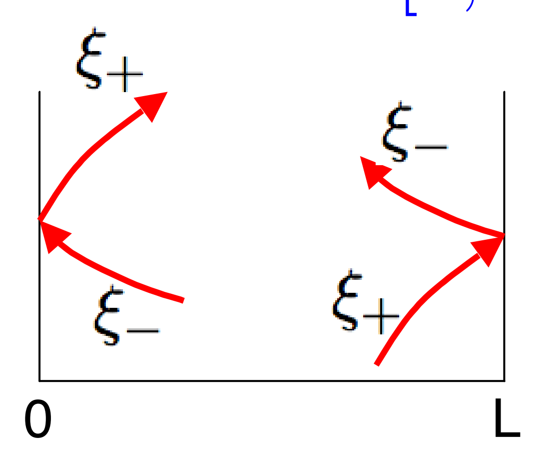
$$\xi_+ = (\xi_1, \xi_2, \dots, \xi_n)$$

$$\xi_- = (\xi_{n+1}, \dots, \xi_{2n})$$

$x \in [0, L]$

Boundary conditions

$$\begin{pmatrix} \xi_+(t,0) \\ \xi_-(t,L) \end{pmatrix} = G \begin{pmatrix} \xi_+(t,L) \\ \xi_-(t,0) \end{pmatrix}$$

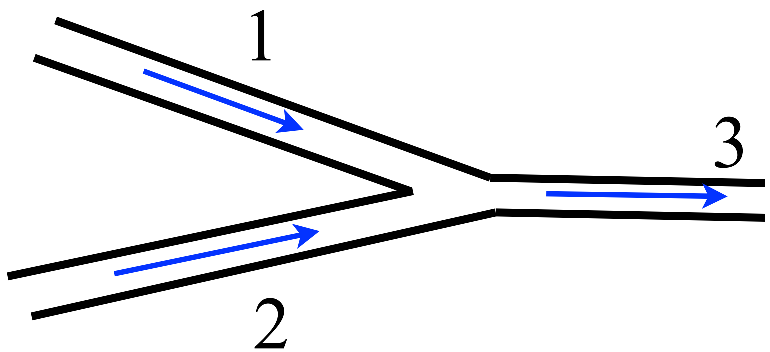


Boundary conditions = Physical constraints

$$\begin{pmatrix} \xi_+(t,0) \\ \xi_-(t,L) \end{pmatrix} = G \begin{pmatrix} \xi_+(t,L) \\ \xi_-(t,0) \end{pmatrix} \iff F(Y(t,0),Y(t,L)) = 0$$

Road traffic

ρ = density
v = velocity

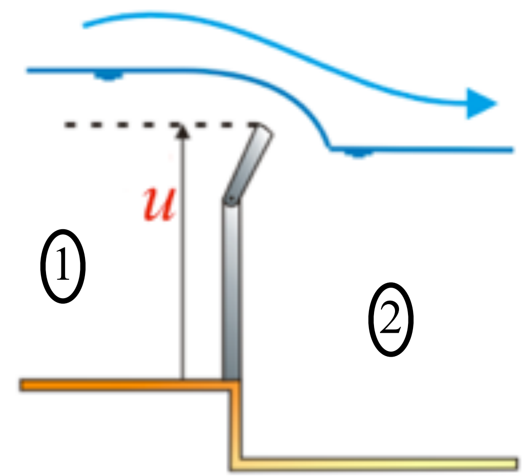


Flow conservation at a junction

$$\rho_3(t,0)v_3(t,0) = \rho_1(t,L)v_1(t,L) + \rho_2(t,L)v_2(t,L)$$

Open channels

h = water depth
v = velocity



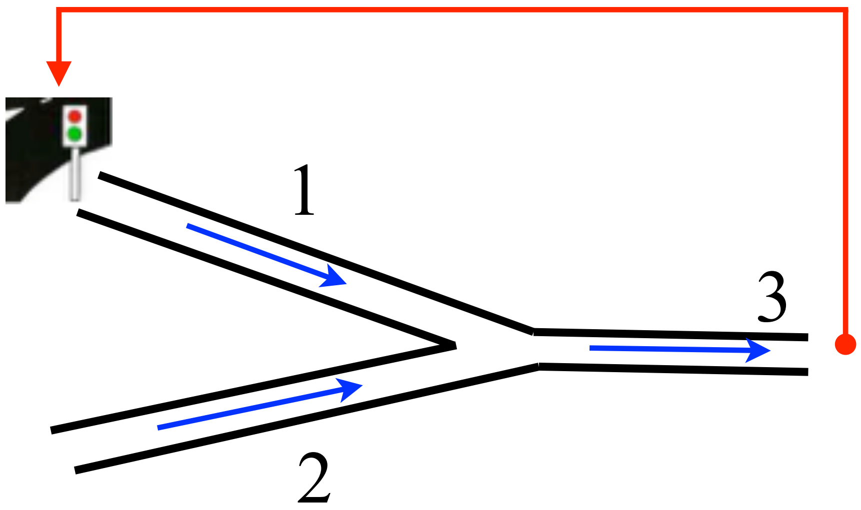
Modelling of hydraulic gates

$$h_2(t,0)v_2(t,0) = \alpha\big(h_1(t,L) - u\big)^{3/2}$$

Boundary conditions = boundary feedback control

Road traffic

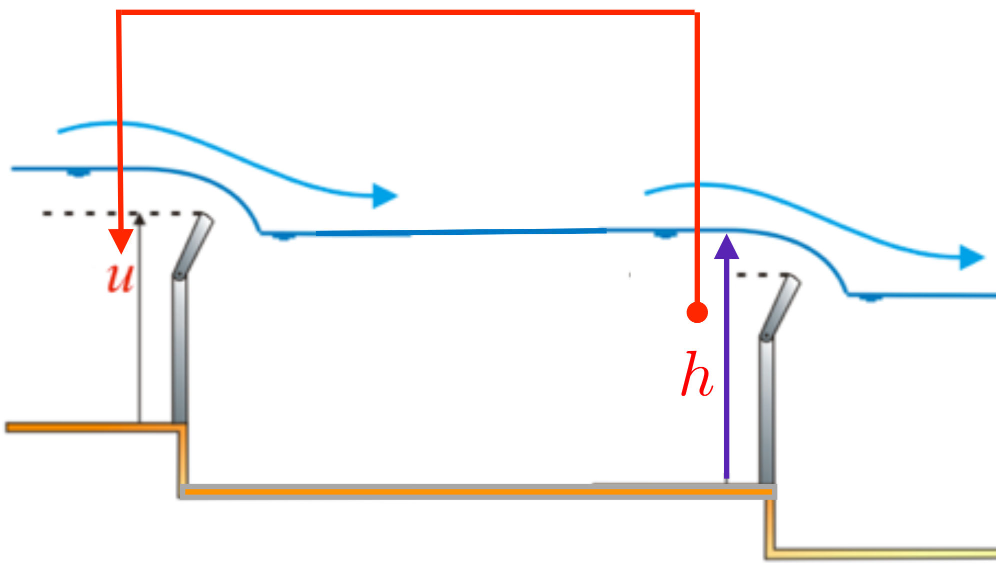
$$\rho = \text{density}$$
$$v = \text{velocity}$$
$$q = \rho v = \text{flux}$$



Feedback implementation of ramp metering

Open channels

$$h = \text{water depth}$$
$$v = \text{velocity}$$



Feedback control of water depth in navigable rivers

G function of the control tuning parameters : How to design the control laws to make the boundary conditions stabilizing ?

$$t \in [0, +\infty) \quad x \in [0, L]$$

$$\partial_t \xi + C(\xi)\partial_x \xi = h(\xi)$$

$$\begin{pmatrix} \xi_+(t,0) \\ \xi_-(t,L) \end{pmatrix} = G \begin{pmatrix} \xi_+(t,L) \\ \xi_-(t,0) \end{pmatrix}$$

$$\xi(0,x) = \xi_0(x)$$

Conditions on
C, h and G
such that $\xi = 0$ is
exponentially stable ?

$$t \in [0, +\infty) \quad x \in [0, L]$$

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Conditions on
C, h and G
such that $\xi = 0$ is
exponentially stable ?

In a linear space χ of functions from $[0, L]$ into $\mathbb{R}^{2n}$

Definition of exponential stability for the norm $\| \ \|_\chi$

$\exists\, \varepsilon, \gamma, \nu$ such that solutions are defined $\forall\, t \in [0, +\infty)$ and

$$\|\xi_0\|_\chi \leqslant \varepsilon \ \Rightarrow \|\xi(t,.)\|_\chi \leqslant \gamma e^{-\nu t} \|\xi_0\|_\chi$$

($+$ usual compatibility conditions adapted to χ)

Lyapunov stability : we start with the linear case

!

System $\partial_t \xi + \Lambda \partial_x \xi = B\xi \quad \Lambda = \text{diag}\{\lambda_i > 0\}$

Boundary condition $\xi(t,0) = K\xi(t,L)$

Lyapunov function $V = \int_0^L \xi^T(t,x) P \xi(t,x) e^{-\mu x} dx$

$\mu > 0 \qquad P = \text{diag}\ \{p_i > 0\}$

$$\dot{V} = -\int_0^L \xi(t,x)\Big(\mu P\Lambda - [B^T P + PB]\Big)\xi(t,x)e^{-\mu x}dx$$

$$-\xi^T(t,L)[P\Lambda e^{-\mu L} - K^T P \Lambda K]\xi(t,L)$$

System	Lyapunov function
$\partial_t \xi + \Lambda \partial_x \xi = B\xi$	$V = \int_0^L \xi^T(t,x) P \xi(t,x) e^{-\mu x} dx$
$\xi(t,0) = K\xi(t,L)$	$\dot{V} = -\int_0^L \xi(t,x)\Big(\mu P \Lambda - [B^T P + PB]\Big)\xi(t,x) e^{-\mu x} dx$
$\xi(0,x) = \xi_0(x)$	$- \xi^T(t,L)[P\Lambda e^{-\mu L} - K^T P \Lambda K]\xi(t,L)$

If $\exists\, P$ (diag. pos.) and $\mu > 0$ such that:

1) $\quad \mu P \Lambda - [B^T P + PB] \succ 0$

2) $\quad P\Lambda e^{-\mu L} - K^T P \Lambda K \succ 0$

($\succ 0$: positive definite)

Then $\exists\, \delta > 0$ s.t. $\dot{V} \leqslant -\delta V \quad \Rightarrow$ exponential stability for L^2-norm

(B-C chapter 5)

Let us now consider the nonlinear case

System $\partial_t \xi + C(\xi)\partial_x \xi = h(\xi)$ $\quad c_i(\xi) > 0$

Boundary condition $\xi(t,0) = G(\xi(t,L))$

Notations: $\Lambda = C(0), \;\; B = h'(0), \;\; K = G'(0)$ (linearization)

If $\exists P$ (diag. pos.) and $\mu > 0$ such that:

1) $\mu P \Lambda - [B^T P + PB] \succ 0$

2) $P \Lambda e^{-\mu L} - K^T P \Lambda K \succ 0$

then the steady-state $\xi \equiv 0$ is exponentially stable for the H^2-norm !

(B-C chapter 6)

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then the steady-state $\xi \equiv 0$ is exponentially stable for the H^2-norm !

Lyapunov function $$V = \int_0^L \left[(\xi^T P \xi + \xi_t P \xi_t + \xi_{tt} P \xi_{tt}) e^{-\mu x} \right] dx$$

(B-C chapter 6)

Special case: a more explicit stability condition

$$\partial_t \xi + C(\xi)\partial_x \xi = h(\xi) \qquad \xi(t,0) = G(\xi(t,L))$$

$$\Lambda = C(0), \quad B = h'(0), \quad K = G'(0)$$

Theorem

If $B = 0$, or if $\|B\|$ sufficiently small,

then $\xi \equiv 0$ is exponentially stable for H^2-norm if

$$\rho_2(K) < 1$$

(boundary damping)

$$\xi(t,0) = K\xi(t,L)$$

$$\rho_2(K) = \min_{\Delta}(\|\Delta K \Delta^{-1}\|, \Delta \text{ positive diagonal})$$

$$(\| \ \| : \text{ 2-norm})$$

(B-C chapters 4 and 6)

Special case: Li Ta Tsien Condition

$$\partial_t \xi + C(\xi)\partial_x \xi = h(\xi) \qquad \xi(t,0) = G(\xi(t,L))$$

$$\Lambda = C(0), \ \ B = h'(0), \ \ K = G'(0)$$

Theorem

If $B = 0$, or if $\|B\|$ sufficiently small,

then $\xi \equiv 0$ is exponentially stable for C^1-norm if !

$$\boxed{\rho(|K|) < 1}$$

$\rho(|K|) = \text{spectral radius of the matrix } \left[|K_{ij}|\right]$

(boundary damping)

$\xi(t,0) = K\xi(t,L)$

(Li Ta Tsien 1994)

(B-C chapter 6)

H^2/C^1 exponential stability

$$\partial_t \xi + C(\xi)\partial_x \xi = h(\xi)$$
$$\xi(t,0) = G(\xi(t,L))$$
$$\Lambda = C(0), \quad B = h'(0), \quad K = G'(0)$$

For every $K \in \mathcal{M}_{m,m}$

$$\rho_2(K) \leqslant \rho(|K|)$$

Example: for $K = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $\rho_2(K) = \sqrt{2} < 2 = \rho(|K|)$

(B-C chapter 4 + Appendix)

For the C^1-norms, the condition $\rho(|K|) < 1$ is sufficient for the stability

the condition $\rho_2(K) < 1$ is not sufficient

(Coron and Nguyen 2015)

H^2/C^1 exponential stability

$$\partial_t \xi + C(\xi)\partial_x \xi = h(\xi)$$
$$\xi(t,0) = G(\xi(t,L))$$
$$\Lambda = C(0), \quad B = h'(0), \quad K = G'(0)$$

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(B-C chapter 4 + Appendix)

For the C^1-norms, the condition $\rho(|K|) < 1$ is sufficient for the stability

the condition $\rho_2(K) < 1$ is not sufficient

There are boundary conditions that are sufficiently damping for the H^2-norm but not for the C^1-norm!

Another special case

$$\partial_t \xi + C(\xi)\partial_x \xi = h(\xi) \qquad\qquad \xi(t,0) = G(\xi(t,L))$$

$$\Lambda = C(0), \;\; B = h'(0), \;\; K = G'(0)$$

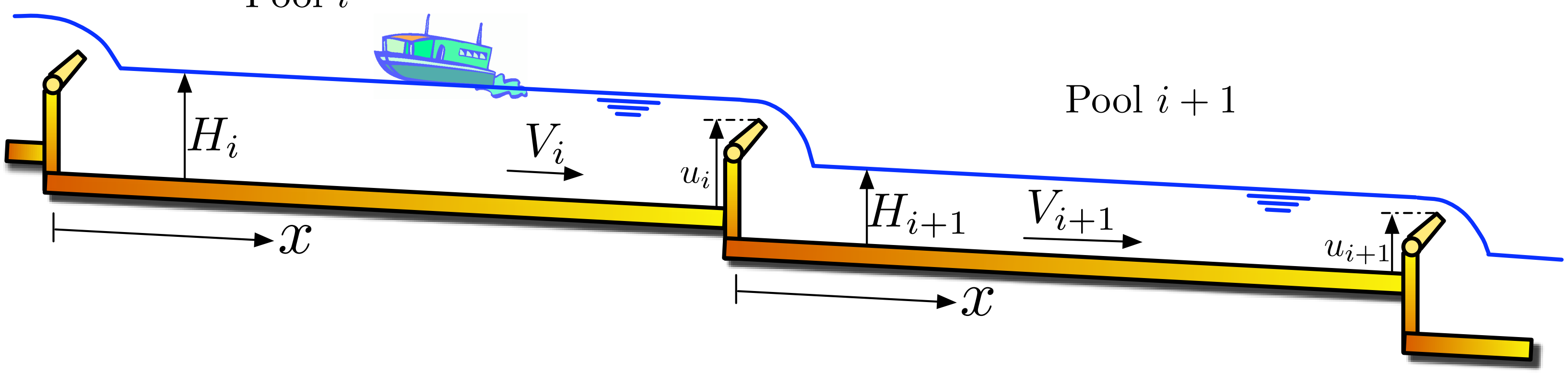
If $\exists\, P$ (diag. pos.) such that:

1) $B^T P + PB \preccurlyeq 0$ (e.g. dissipative friction term)

2) $\rho_2(K) = (\|\Delta K \Delta^{-1}\| < 1$ with $\Delta^2 = P\Lambda$ (boundary damping)

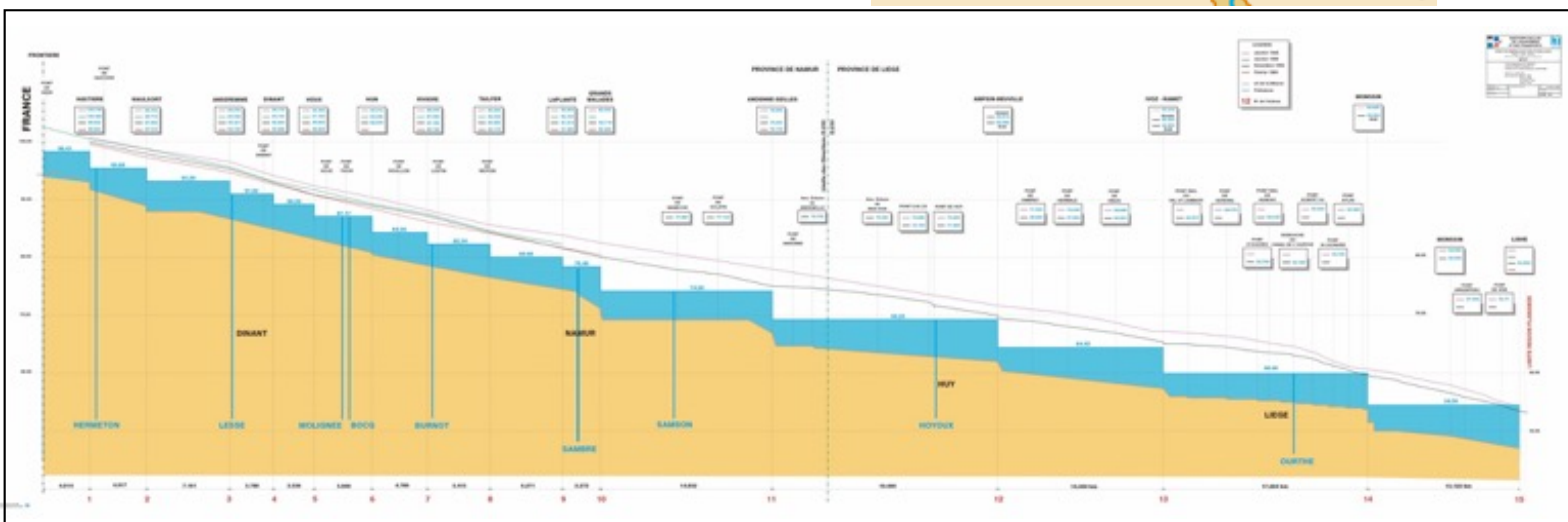
The stability conditions hold also, with appropriate modifications, for systems with positive and negative characteristic velocities $c_i(\xi)$, as we shall see in the example.

Example: Boundary control for a channel with multiple pools

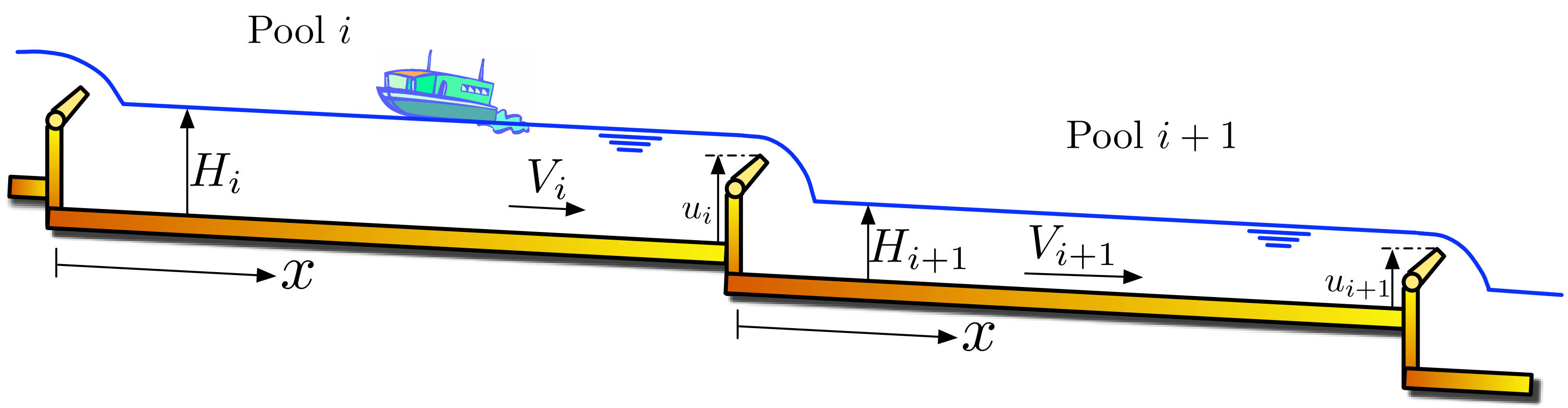


Meuse river (Belgium)

Real-life application : level control in navigable rivers.
Meuse river (Belgium).



Model : Saint-Venant equations



$$\partial_t \begin{pmatrix} H_i \\ V_i \end{pmatrix} + \partial_x \begin{pmatrix} H_i V_i \\ \frac{1}{2} V_i^2 + g H_i \end{pmatrix} = \begin{pmatrix} 0 \\ g[S_i - C_i V_i^2 H_i^{-1}] \end{pmatrix}, \qquad i = 1 \ldots, n.$$

slope friction

$(\text{width} = W)$

Boundary conditions

1) Conservation of flows $\quad H_i(t,L)V_i(t,L) = H_{i+1}(t,0)V_{i+1}(t,0) \quad i = 1, \ldots, n-1$

2) Gate models $\quad H_i(t,L)V_i(t,L) = k_G \sqrt{\left[H_i(t,L) - u_i(t)\right]^3} \quad i = 1, \ldots, n$

3) Input flow $\quad W H_1(t,0)V_1(t,0) = Q_0(t)$ = disturbance input in navigable rivers or control input in irrigation networks

Boundary control design

$(H_i^*, V_i^*) = \text{steady-state}$

Riemann coordinates

$$\xi_i = (V_i - V_i^*) + (H_i - H_i^*)\sqrt{\frac{g}{H_i^*}} \qquad \xi_{n+i} = (V_i - V_i^*) - (H_i - H_i^*)\sqrt{\frac{g}{H_i^*}} \qquad i = 1, \ldots, n.$$

Control objective

$$\xi_{n+i}(t, L) = -k_i \xi_i(t, L) \qquad i = 1, \ldots, n$$

k_i = control tuning parameters

Gate model

$$H_i(t, L)V_i(t, L) = k_G \sqrt{\left[H_i(t, L) - u_i(t)\right]^3} \qquad i = 1, \ldots, n$$

H_i^* = level set points

Control law

$$u_i(t) = H_i(t, L) - \left[\frac{H_i(t, L)}{k_G}\left(\frac{1 - k_i}{1 + k_i}(H_i(t, L) - H_i^*)\sqrt{\frac{g}{H_i^*}} + \sqrt{\frac{S_i H_i^*}{C}}\right)\right]^{2/3} \qquad i = 1, \ldots, n$$

Linearized Saint-Venant equations in Riemann coordinates

$$i = 1, \ldots, n$$

$$\partial_t \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} + \partial_x \begin{pmatrix} \lambda_i & 0 \\ 0 & -\lambda_{n+i} \end{pmatrix} \partial_x \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} = - \begin{pmatrix} \gamma_i & \delta_i \\ \gamma_i & \delta_i \end{pmatrix} \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix}$$

$$-\lambda_{n+i} = V_i^* - \sqrt{gH_i^*} \quad < \quad 0 \quad < \quad \lambda_i = V_i^* + \sqrt{gH_i^*} \qquad\qquad 0 \quad < \quad \lambda_{n+i} \quad < \quad \lambda_i$$

$$0 \quad < \quad \gamma_i = gS_i \left(\frac{1}{V_i^*} - \frac{1}{2\sqrt{gH_i^*}} \right) \quad < \quad \delta_i = gS_i \left(\frac{1}{V_i^*} + \frac{1}{2\sqrt{gH_i^*}} \right)$$

Lyapunov stability

$$\partial_t \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} + \partial_x \begin{pmatrix} \lambda_i & 0 \\ 0 & -\lambda_{n+i} \end{pmatrix} \partial_x \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix} = - \begin{pmatrix} \gamma_i & \delta_i \\ \gamma_i & \delta_i \end{pmatrix} \begin{pmatrix} \xi_i \\ \xi_{n+i} \end{pmatrix}$$

Subcritical flow condition $\Rightarrow$ $0 < \lambda_{n+i} < \lambda_i \qquad 0 < \gamma_i < \delta_i$

$$V = \sum_{i=1}^{n} \int_0^L \left(p_i \xi_i^2 e^{-\mu x} + p_{n+i} \xi_{n+i}^2 e^{\mu x} \right) dx, \qquad p_i, p_{n+i}, \mu > 0$$

If parameters p_i selected such that $\quad p_i \gamma_i = p_{n+i} \delta_i \qquad p_{i+1} = \varepsilon p_i$

(dissipative friction)

ε and μ sufficiently small

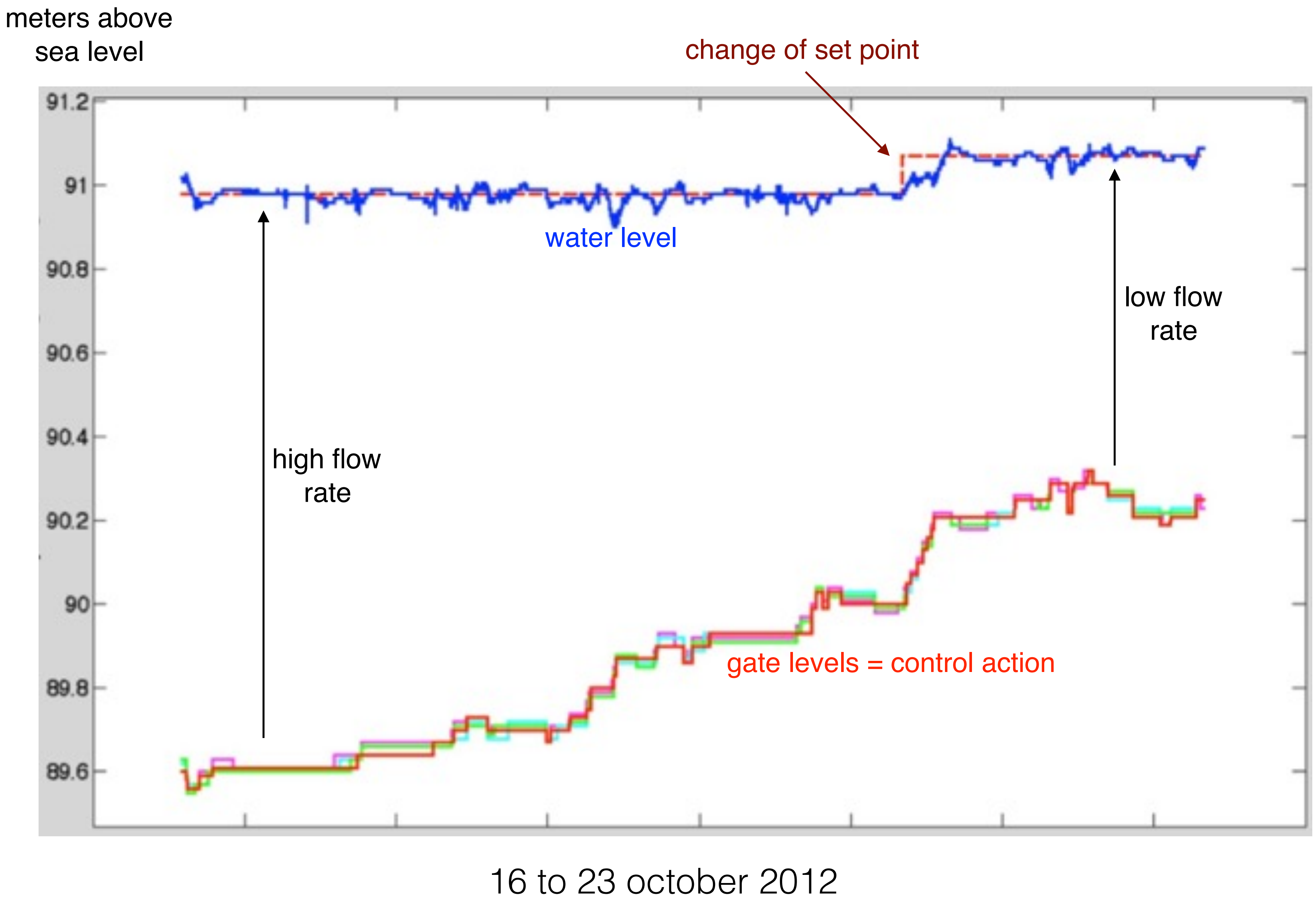
control tuning parameters k_i such that

$$k_i^2 \frac{\delta_i \lambda_{n+i}}{\gamma_i \lambda_i} < 1 \qquad \text{(boundary damping)}$$

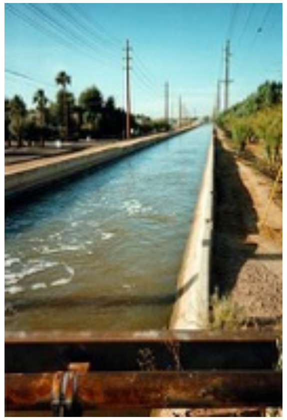
$\Rightarrow$ **steady-state exponentially stable**

(B-C chapter 5)

Experimental result (Dinant)



The nonuniform case : example



Now we consider a pool of a prismatic **horizontal** open channel with a rectangular cross section and a unit width.

$H(x,t)$ = water depth

$V(x,t)$ = water velocity

C = friction coefficient

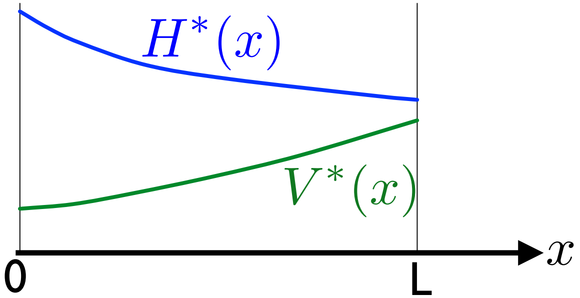
$$\partial_t H + \partial_x(HV) = 0$$

$$\partial_t V + \partial_x\left(\frac{V^2}{2} + gH\right) + gC\frac{V^2}{H} = 0$$

Steady-state $H^*(x), V^*(x)$ for a constant flow rate $Q^* = H^*(x)V^*(x)$

$$\frac{dV^*}{dx} = \frac{gC}{Q^*}\left(\frac{(V^*(x))^5}{gQ^* - (V^*(x))^3}\right)$$

Nonuniform steady state



$$\partial_t H + \partial_x (HV) = 0$$
$$\partial_t V + \partial_x \left(\frac{V^2}{2} + gH \right) + gC \frac{V^2}{H} = 0$$

Linearization about the steady-state

$$h(t,x) := H(t,x) - H^*(x), \quad v(t,x) := V(t,x) - V^*(x)$$

Linearized model in physical coordinates

$$\begin{pmatrix} h_t \\ v_t \end{pmatrix} + \begin{pmatrix} V^* & H^* \\ g & V^* \end{pmatrix} \begin{pmatrix} h_x \\ v_x \end{pmatrix} + \begin{pmatrix} V_x^* & H_x^* \\ -gC\frac{V^{*2}}{H^*} & V_x^* + 2gC\frac{V^*}{H^*} \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix} = 0$$

$$\begin{pmatrix} h_t \\ v_t \end{pmatrix} + \begin{pmatrix} V^* & H^* \\ g & V^* \end{pmatrix} \begin{pmatrix} h_x \\ v_x \end{pmatrix} + \begin{pmatrix} V_x^* & H_x^* \\ -gC\frac{V^{*2}}{H^*} & V_x^* + 2gC\frac{V^*}{H^*} \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix} = 0$$

Lyapunov function

$$\mathbf{V} = \int_0^L \left(gh^2 + H^* v^2\right) dx = \int_0^L \begin{pmatrix} h & v \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & H^* \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix} dx$$

$$\frac{d\mathbf{V}}{dt} = -\int_0^L \left(Y^T N(x) Y\right) dx - \left[Y^T M(x) Y\right]_0^L$$

$$Y = \begin{pmatrix} h \\ v \end{pmatrix} \qquad N(x) = \begin{pmatrix} \frac{g^2 C V^{*3}}{H^*(gH^* - V^{*2})} & -\frac{gCV^{*2}}{H^*} \\ -\frac{gCV^{*2}}{H^*} & \frac{2gCV^{*3}}{(gH^* - V^{*2})} + 4gCV^* \end{pmatrix} \qquad M(x) = \begin{pmatrix} gV^* & gH^* \\ gH^* & H^* V^* \end{pmatrix}$$

$$\begin{pmatrix} h_t \\ v_t \end{pmatrix} + \begin{pmatrix} V^* & H^* \\ g & V^* \end{pmatrix} \begin{pmatrix} h_x \\ v_x \end{pmatrix} + \begin{pmatrix} V_x^* & H_x^* \\ -gC\frac{V^{*2}}{H^*} & V_x^* + 2gC\frac{V^*}{H^*} \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix} = 0$$

Subcritical flow (i.e. fluvial): $gH^* - V^{*2} > 0$

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positive definite

$$\begin{pmatrix} h_t \\ v_t \end{pmatrix} + \begin{pmatrix} V^* & H^* \\ g & V^* \end{pmatrix} \begin{pmatrix} h_x \\ v_x \end{pmatrix} + \begin{pmatrix} V_x^* & H_x^* \\ -gC\frac{V^{*2}}{H^*} & V_x^* + 2gC\frac{V^*}{H^*} \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix} = 0$$

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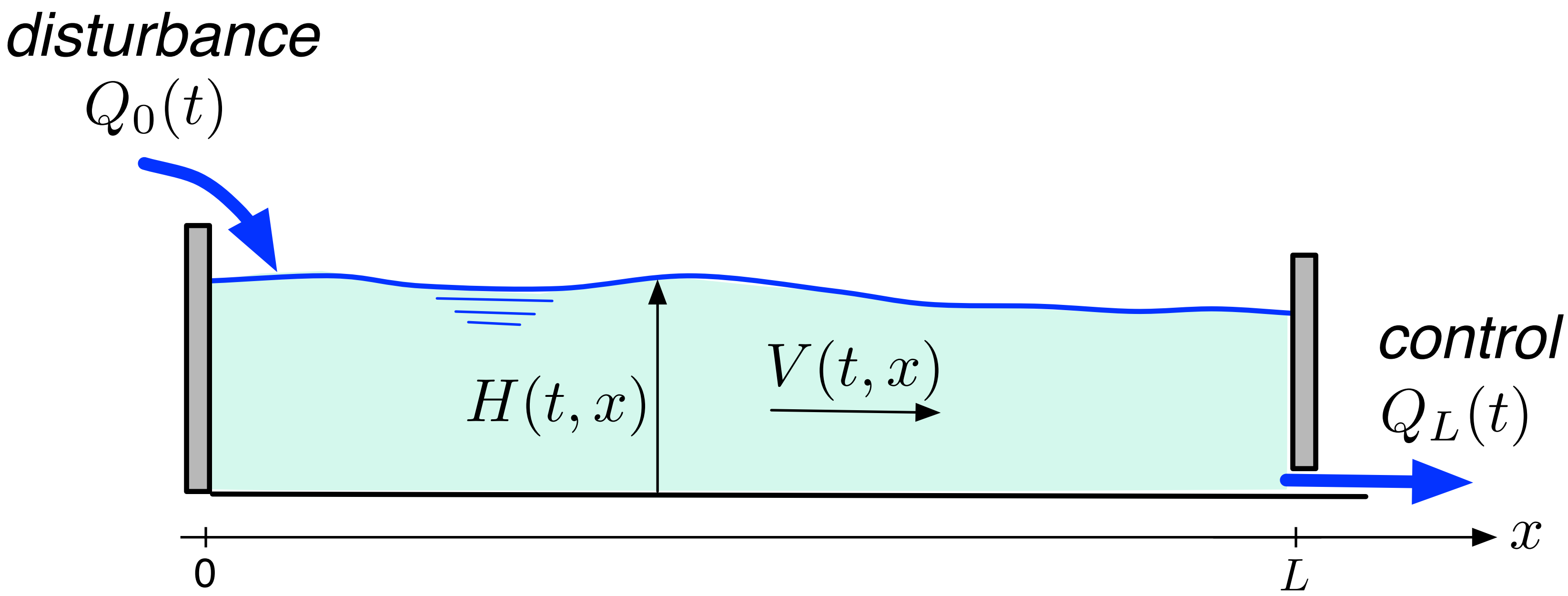
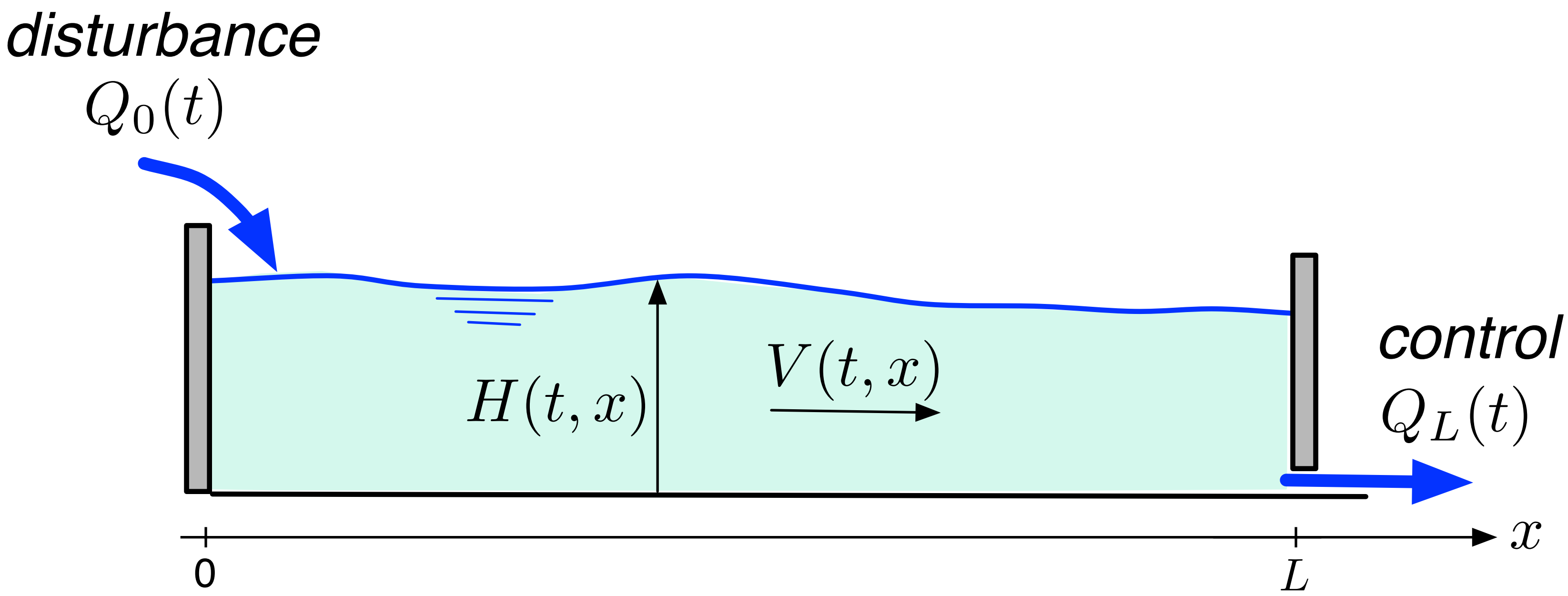
boundary damping conditions ?

$$Y = \begin{pmatrix} h \\ v \end{pmatrix} \qquad N(x) = \begin{pmatrix} \frac{g^2 C V^{*3}}{H^*(gH^* - V^{*2})} & -\frac{gCV^{*2}}{H^*} \\ -\frac{gCV^{*2}}{H^*} & \frac{2gCV^{*3}}{(gH^* - V^{*2})} + 4gCV^* \end{pmatrix} \qquad M(x) = \begin{pmatrix} gV^* & gH^* \\ gH^* & H^*V^* \end{pmatrix}$$

positive definite

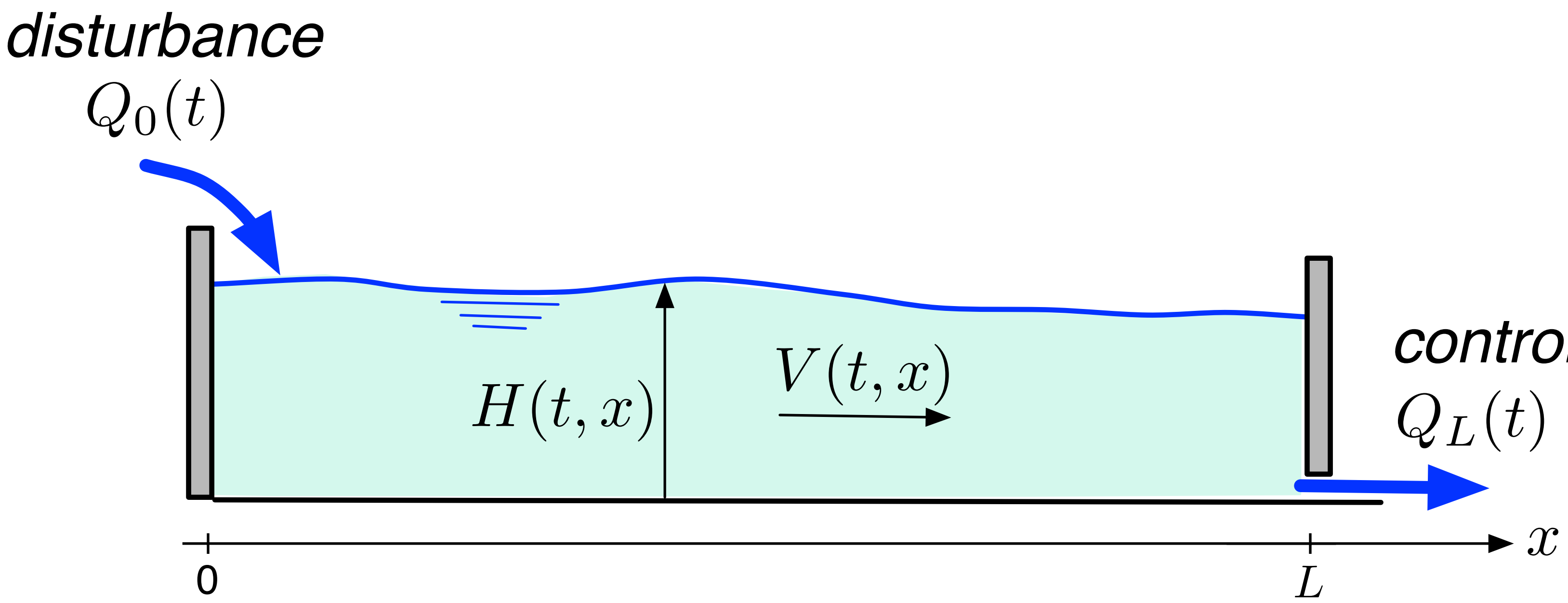


control $$Q_L(t) = \underbrace{Q_0(t)}_{\text{feedforward}} + \underbrace{k_P(H(t,L) - H_{sp})}_{\text{feedback}}$$



measured / measured / set point / control tuning

control $$Q_L(t) = Q_0(t) + k_P(H(t,L) - H_{sp})$$

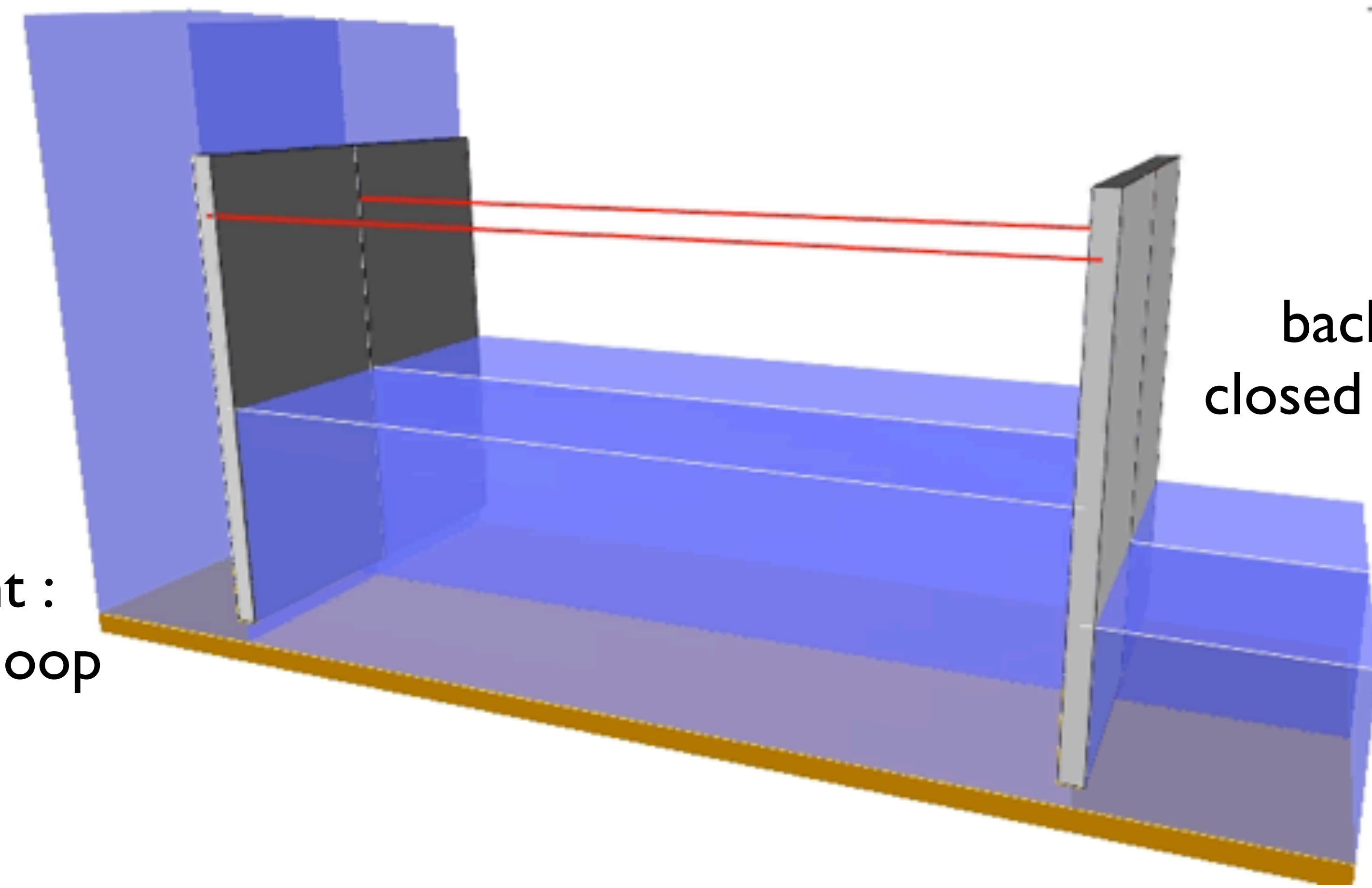


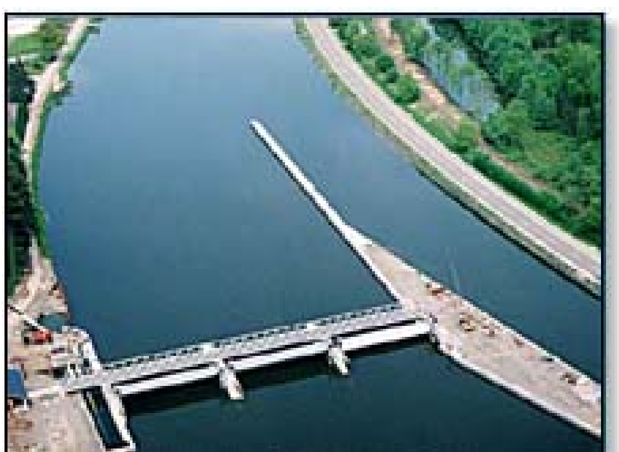
control $\quad Q_L(t) = Q_0(t) + k_P(H(t,L) - H_{sp})$

boundary damping $\quad \left[Y^T M(x) Y\right]_0^L > 0$

$$\Longrightarrow \quad \frac{V^*(0)}{H^*(0)}(gH^*(0) - V^{*2}(0))h^2(t,0) + \left[\frac{gH^*(L) - V^{*2}(L)}{H^*(L)}(2k_P - V^*(L)) + \frac{V^*(L)}{H^*(L)}k_P^2\right] h^2(t,L) \; > 0$$

$\Longrightarrow$ exponential stability if $|k_P|$ is sufficiently large ...





Meuse
river

Thank you !